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CARTAN-KAHLER THEORY AND PROLONGATION

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ABSTRACT
In this paper we speak about Cartan-Kahler Theory and Applications to exterior differential system, we have shown that some definitions, theorems and examples. Therefore, the results are derived from the proprieties of Cartan-Kahler theory and Applications to exterior differential system, with the correspondence results that derived heuristically by many authors.

KEYWORDS: Cartan-Kahler Theory, exterior differential system and prolongation.

INTRODUCTION
In this paper, we focused to discuss how problems in differential geometry and partial differential equations can often be reformed as problems about integral manifolds of appropriate exterior differential systems. More-over, in differential geometry, particularly in the theory and applications of the moving frame and Cartan's methods of the equivalence, the problems to be studied often appear naturally in the form of exterior differential system anyway.
This motivates the problem of finding a general method of constructing integral manifold. When the exterior differential system I has a particularly simple form, standard differential calculus and the techniques of ordinary differential equations allow a complete (local) description of the integral manifolds of I.

KÄHLER-ORDINARY AND KÄHLER-REGULAR INTEGRAL ELEMENTS
Let Ω a differential n-form on a m-dimensional manifold M. Let $G_n(TM,\Omega)=\{E\in G_n(TM)|\Omega_{\mathrm{E}}\neq 0\}$, where $G_n(TM)$ is the Grassmanian of TM, i.e., the set of n-dimensional subspace of TM. We denote by $\mathcal{V}_n(I,\Omega)=\mathcal{V}_n(I)\cap G_n(TM,\Omega)$ the set of integral elements of I on which $\Omega_E\neq 0$.

Definition (2.1):
An integral element $E\in\mathcal{V}_n(I)$ is called *Kähler-ordinary* if there exists a differential n-form Ω such that $\Omega_E\neq 0$. Moreover, if the function r is locally constant in some neighborhood of E, then is said *Kähler-regular*.

Example (1):
We will show that all of the 2-dimensional integral elements of I are Kähler-regular. Let $\Omega=dx^4\wedge dx^5$. Then every element $E\in G_2(T\mathbb{R}^5,\Omega)$ has a basis$\{X_4,X_5\}$ of the form

$$\begin{aligned} X_4(E)&=\partial/\partial x^4+p_4^1(E)\,\partial/\partial x^1+p_4^2(E)\,\partial/\partial x^2+p_4^3(E)\,\partial/\partial x^3\\ X_5(E)&=\partial/\partial x^5+p_5^1(E)\,\partial/\partial x^1+p_5^2(E)\,\partial/\partial x^2+p_5^3(E)\,\partial/\partial x^3. \end{aligned}\quad(1)$$

the functions $x^1,\dots,x^5,p_5^1,\dots,p_5^3$ for a coordinate system on $G_2(T\mathbb{R}^5,\Omega)$.Computation gives

$$\begin{gathered}(\vartheta^1\wedge dx^4)\Omega=-p_5^1\\ (\vartheta^1\wedge dx^5)\Omega=p_4^1+(x^3-x^4x^5)\\ (\vartheta^2\wedge dx^4)\Omega=-p_5^2-(x^3+x^4x^5)\\ (\vartheta^2\wedge dx^5)\Omega=p_4^2\\ (\vartheta^3\wedge dx^4)\Omega=-p_5^3+x^4\\ (\vartheta^3\wedge dx^5)\Omega=-p_4^3-x^5\end{gathered}\quad(2)$$

every point of $\mathcal{V}_2(I)$ is kähler-ordinary. Since none of these elements has any extension to a 3-dimensional integral element, it follows that $r(E) = -1, \forall\, E \in \mathcal{V}_2(I)$. Thus, every element of $\mathcal{V}_2(I)$ is also kähler-regular.

INTEGRAL FLAGS, ORDINARY AND REGULAR INTEGRAL ELEMENTS

Definition (3.1)

An integral flag of I on $z \in M$ of length n is a sequence of integral elements $E_k$of I: $(0)_z \subset E_1 \subset \cdots \subset E_n \subset T_z M$.

Definition (3.2)

Let I be an exterior differential system on M. An integral element $E \in \mathcal{V}(I)$ is said ordinary if its base point $z \in M$ is an ordinary zero of $I_0 = I \cap A^0(M)$ and if there exists an integral flag$(0)_z \subset E_1 \subset \cdots \subset E_n = E \subset T_z M$ where the $E_k, k \leq n-1$,are Kähler-regular integral elements. Moreover, if E is Kähler-regular, then E is said regular.

Theorem (3.2): (Cartan's Test)

Let $I \subset A^*(M)$ be an exterior ideal which doesn't contain 0-forms (functions on M). Let $(0)_z \subset E_1 \subset \cdots \subset E_n \subset T_z M$ be an integral flag of I. For any $k < n$, we denote by c_k the codimension of the polar space $H(E_k)$ in $T_z M$. Then $\mathcal{V}_n(I) \subset G_n(TM)$ is at least of $c_0, c_1, \dots, c_{n-1}$ codimension at E_n. Moreover, E_n is an ordinary integral flag if and only if E_n has a neighborhood U in $G_n(TM)$ such that $\mathcal{V}_n(I) \cap U$ is a manifold of $c_0 + c_1 + \cdots + c_{n-1}$ codimension in U.

Example (2): Using theorem (3.2)

We can give a quick proof the none-elements in $\mathcal{V}_2(I)$ are ordinary. For any integral flag $(0)_z \subset E_1 \subset E_2 \subset T_z\mathbb{R}^5$, we know that $c_0 \leq 2$ since there are two independent 1-forms in I. Also, since $E_2 \subset H(E_1)$, it follows that $c_1 \leq 3$. Since there is unique 2-dimensional integral element at each point of $\mathbb{R}^5$ it follows that $\mathcal{V}_2(I)$ has codimension six in $G_2(T\mathbb{R}^5)$. Since $c_0 + c_1 < 6$, it follows, by theorem (3.2), that none of the integral flags of length two can be ordinary. Hence there are no ordinary integral elements of dimension two.

Example (3)

Let $M = \mathbb{R}^6$ with coordinates $x^1, x^2, x^3, u^1, u^2, u^3$. Let I be the differential system generated by the 2-form

$$\vartheta = d(u_1 dx^1 + u_2 dx^2 + u_3 dx^3) - (u_1 dx^2 \wedge dx^3 + u_2 dx^3 \wedge dx^1 + u_3 dx^1 \wedge dx^2). \tag{3}$$

Of course, I generated algebraically by the forms $\{\vartheta, d\vartheta\}$. Then, we have

$$d\vartheta = -(du_1 \wedge dx^2 \wedge dx^3 + du_2 \wedge dx^3 \wedge dx^1 + du_3 \wedge dx^1 \wedge dx^2). \tag{4}$$

We can use the theorem (3.2) to show that all of 3-dimensional integral elements of I on which $\Omega = dx^1 \wedge dx^2 \wedge dx^3$ does not vanish are ordinary. Let $E \in \mathcal{V}_3(I, \Omega)$ be fixed with base point $z \in \mathbb{R}^6$. Let (e_1, e_2, e_3) be basis of E which is dual of basis $(dx^1, dx^2, dx^3)$of E^*. Let E_1 be the line spanned by e_1, let E_2 be the 2-plane spanned by the pair $\{e_1, e_2\}$, and E_3 be E. Then $(0)_z \subset E_1 \subset E_2 \subset E_3$ is an integral flag. Since I is generated by $\{\vartheta, d\vartheta\}$, it follows that $c_0 = 0$. Moreover, since $\vartheta(v, e_1) = \tau_1(v)$where$\tau_1 \equiv du_1 \text{mod}(dx^1, dx^2, dx^3)$, it follows that $c_1 = 1$. Note that, since $E_3 \subset H(E_2)$, it follows that $c_2 \leq 3$. On the hand, we have the formula

$$\begin{aligned} \vartheta(v, e_1) &= \tau_1(v) \\ \vartheta(v, e_2) &= \tau_2(v) \\ d\vartheta(v, e_1, e_2) &= -\tau_3(v) \end{aligned} \tag{5}$$

where in each case, $\tau_k \equiv du_k \mod(dx^1, dx^2, dx^3)$. Since the 1-form τ_k are clearly independent and annihilate $H(E_2)$, it follows that $c_2 \geq 3$. Combined with the pervious argument, we have $c_2 = 3$. It follows by theorem (3.2) that the codimension of $\mathcal{V}_3(I)$ in $G_3(T\mathbb{R}^6)$ at E is at least $c_0 + c_1 + c_3 = 4$. Now, we shall illustrate that $\mathcal{V}_3(I, \Omega)$ is smooth submanifold of $G_3(T\mathbb{R}^6)$ of codimension four, and thence, by theorem (3.2), conclude that E is ordinary. To do this, we introduce functions p_{ij} on $G_3(T\mathbb{R}^6, \Omega)$ with the property that, for each $E \in G_3(T\mathbb{R}^6, \Omega)$ based at $z \in \mathbb{R}^6$, the forms $\tau_i = du_i - p_{ij}(E)\, dx^j \in T_z^*(\mathbb{R}^6)$ are a basis for the 1-forms which annihilate E. Then the functions (x, u, p) form a coordinate system on $G_3(T\mathbb{R}^6, \Omega)$. It is easy to compute that

$$\vartheta_E = (p_{23} - p_{32} - u_1) dx^2 \wedge dx^3 + (p_{31} - p_{13} - u_2) dx^3 \wedge dx^1 + (p_{12} - p_{21} - u_3) dx^1 \wedge dx^2 \tag{6}$$

$$d\vartheta_E = -(p_{11} + p_{22} + p_{33}) dx^1 \wedge dx^2 \wedge dx^3. \tag{7}$$

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It follows that the condition that $E \in G_3(T\mathbb{R}^6, \Omega)$ be an integral element of I is equivalent to the vanishing of four functions on $G_3(T\mathbb{R}^6, \Omega)$ whose differentials are independent. Thus, $\mathcal{V}_3(I, \Omega)$ is smooth manifold of codimension 4 in $G_3(T\mathbb{R}^6, \Omega)$.

Proposition (3.3):
Let $I \cap A^*(M)$ an exterior ideal which don't contains 0-forms. Let $E \subset \mathcal{V}_n(I)$ be an integral element of I at the point $z \in M$. Let $\omega_1, \omega_2, \dots, \omega_n, \tau_1, \tau_2, \dots, \tau_s$(where $s = \dim M - n$) be a coframe in an open neighborhood of $z \in M$ such that $E = \{v \in T_z M | \tau_a(v) = 0, \quad \forall a = 1, \dots, s\}$. For all $p \le n$, we define $E_p = \{v \in E | \omega_k(v) = 0, \ \forall\, k > p\}$. Let $\{\varphi_1, \dots, \varphi_r\}$ be the set differential forms which generate the exterior ideal I, where φ_ρ is of $\left(d_\rho + 1\right)$ degree. For all ρ, there exists an expansion

$$\varphi_\rho = \sum_{|J| = d_\rho} \tau_\rho^J \wedge \omega_J + \tilde{\varphi}_\rho \tag{8}$$

where the 1-forms τ_ρ^J are linear combinations of theτ's and the terms $\tilde{\varphi}_\rho$ are, either of degree 2 or more in theτ's or else vanish at z.
Moreover, we have the formula

$$H\left(E_p\right) = \{v \in T_z M | \tau_a(v) = 0, \quad \forall\, \rho \text{ and } \sup J \le p\} \tag{9}$$

In particular, for the integral flag $(0)_z \subset E_1 \subset \cdots \subset E_n \cap T_z M$ of I, c_p is the number of the linear independent forms $\left\{\tau_\rho^J\big|_z \text{such that} \sup J \le p\right\}$.

Theorem (3.4):(Cartan-Kähler)
Let $I \subset \Lambda^*(M)$ be a real analytic exterior differential ideal. Let $P \subset M$ a p-dimensional connected real analytic Kähler-Regular integral manifold of I. Suppose that $r = r(P) \ge 0$. Let $R \subset M$ be a real analytic submanifold of M of codimension r which contains P and such that $T_x R$ and $H(T_x P)$ are transversals in $T_x M, \forall\, x \in P \subset M$.
Then, there exists a $(p+1)$-dimensional connected real analytic integral manifold $X$of I, such that $P \subset X \subset R$. $X$is unique in the sense that another integral manifold of I having the stated properties, coincides with X on an open neighborhood of P.

STATEMENT AND PROOF OF CARTAN-KÄHLER

The Cartan-kähler theorem depends on the fundamental existence theorem of Cauchy and Kowalevski dealing with differential equations, and Cauchy-Kowalevski theorem uses the power series method. Consequently, Cartan-kähler theory is a real-analytic and local theory. This theorem is a coordinate-free, geometric generalization of the classical Cauchy-Kowalevski theorem, which we state presently.
We shall use the index ranges $1 \le i, j \le n$ and $1 \le a, b \le s$.

Corollary (4.1): (Cartan-Kähler)
Let I be an analytic exterior differential on a manifold M. If $E \subset T_z M$ is an ordinary integral element of I, there exists an integral manifold of I passing through z and having E as a tangent space at the point z.

Example (4):
As a simple illustration of the Cartan-kähler theorem we consider the partial differential equation in the unknown function $u(x, y)$ given by

$$\partial^2 u / \partial y^2 = \partial u / \partial x \tag{10}$$

on$\mathbb{R}^6 = \{(x, y, u, p, q, r)\}$ we put

$$\begin{aligned} \vartheta_1 &= du - p\, dx - q\, dy \\ \vartheta_2 &= du - p\, dy - r\, dx \end{aligned} \tag{11}$$

then the above partial differential equation translated to the closed exterior differential system on $M = \mathbb{R}^6$ given by

$$\Lambda = \{\vartheta_1, \vartheta_2, \omega_1 = dp \wedge dx + dq \wedge dy, \omega_2 = dp \wedge dy + dr \wedge dx\}. \tag{12}$$

Since $\vartheta_1 \wedge \vartheta_2 \ne 0$ then, every point is a regular integral point, and $s_0 = 2$. Fix an origin, $0 = (0, 0, u_0, p_0, q_0, r_0) \in M$. To find an integral sub manifold

$$g(x) = \left(x, 0, \Phi^1(x), \dots, \Phi^4(x)\right) \tag{13}$$

we must solve

$$\partial u/\partial x = p, \quad d\Phi^1/dx = \Phi^2, \quad \partial q/\partial x = r, \quad d\Phi^3/dx = \Phi^4 \tag{14}$$

with $u = u_0, q = q_0$ at $x = 0$. Let Φ^1 and Φ^3 be arbitrary real-analytic functions in x such that $\Phi^1(0) = u_0$ and $\Phi^3(0) = q_0$. (We can specify u and $\partial u/\partial u$ along $y = 0$). Now, we have $g(x)$. By taking a tangent vector

$$v = (1, 0, v_u, v_p, v_q, v_r) \in M_0. \tag{15}$$

Then the equations $\vartheta_1(v) = \vartheta_2(v) = 0$ implies $v_u = p_0, v_q = r_0$. The dual polar space of $E_0^1 = [v]$ is spanned by ϑ_1, ϑ_2 and the two 1-forms

$$i_v\omega_1 = v_p\, dx - dp + r_0\, dy \quad i_v\omega_2 = v_r\, dx - dr + r_p\, dy \tag{16}$$

the polar matrix of these four 1-forms with respect to (dx, dy, du, dp, dq, dr) is

$$\begin{bmatrix} p_0 & q_0 & 1 & 0 & 0 & 0 \\ r_0 & p_0 & 0 & 0 & -1 & 0 \\ v_p & r_0 & 0 & -1 & 0 & 0 \\ v_r & v_p & 0 & 0 & 0 & -1 \end{bmatrix} \tag{17}$$

this matrix has rank four for any quadruple (p_0, r_0, v_p, v_r). In particular,

$$s_0 + s_1 = 4, \quad s_1 = 2,$$

and $\sigma_2 = 0$. Therefore, there exists a unique solution

$$f(x, y) = (x, y, F^1(x, y), \ldots, F^4(x, y)), \quad \text{with } f(x, 0) = g(x). \tag{18}$$

PROLONGATION

Roughly speaking, the prolongations of a differential system are the differential system obtained by adjoining to the original differential system its differential consequences. The concept of *prolongation tower*, which will be defined below, gives an abstract formulation of the operation of the prolongation. A general conjecture of ElieCartan, [2], proved by Kuranishi, [3], for a wide class of differential systems, state that an analytic differential system with independence condition it's takes a finite number of prolongations for it to be either involutive or incompatible, or has no solutions. This result is known as Cartan-Kuranishi Theorem. The proof of Cartan's conjecture has been given under a different set of hypotheses in the treatise [1]. Our purpose is to review some of the basic aspects of the prolongation theorem. We assume that all manifolds and the differential systems under consideration are of class C^ω. The prolongation tower of an exterior differential system with independence condition (I, Ω) on an n-dimensional manifold M is defined as a follows. Let $f: W_p \to M$ be an immersion and let $f_*: W_p \to G_p(M)$ denote the map into the Grassmann bundle of p-planes in TM determined by f. The Grassmann bundle $G_p(M)$ is endowed with a canonical exterior differential system $C^{(1)}$ defined the property that $f_*{}^*C^{(1)} = 0$ for any immersion $f: W_p \to M$. Using affine fiber coordinates $(x^i, u^\alpha, u_i^\alpha)$, $1 \le i \le p$, $1 \le \alpha \le n$, on Grassmann bundle $G_p(M)$, the system $C^{(1)}$ is defined as the differential ideal generated by the 1-form

$$\vartheta^\alpha = du^\alpha - \sum_{i=1}^{p} u_i^\alpha dx^i. \tag{19}$$

We choose component $V_p(I)$ of the sub-variety of $G_p(M)$ defined by the p-dimensional admissible integral elements of I and assume $V_p(I)$ to be C^ω manifold.

Definition (5.1):

The first prolongation of I is the exterior differential system $I^{(1)}$ defined by

$$I^{(1)} = C^{(1)}\big|_{V_p(I)} \tag{20}$$

For notational simplicity, we use the notation M^1 to denote the $V_p(I)$. We also assume that the map $\pi^{1,0}: (M^{(1)}, I^{(1)}) \to (M, I)$ is a C^ω submersion. The prolongation tower of I is then defined by induction,

$$\cdots \xrightarrow{\pi^{k+1,k}} (M^{(k)}, I^{(k)}) \xrightarrow{\pi^{k,k-1}} \cdots \xrightarrow{\pi^{2,1}} (M^{(1)}, I^{(1)}) \xrightarrow{\pi^{1,0}} (M, I). \tag{21}$$

The infinite prolongation $(M^{(\infty)}, I^{(\infty)})$ of (M, I) is then defined as the inverse limit of this tower

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$$M^{(\infty)} := \lim_{\leftarrow} M^{(k)}, \qquad I^{(\infty)} = \bigcup_{k\geq 0} I^{(k)}. \qquad (22)$$

We now present a statement of prolongation theorem as given in [7]

Theorem (5.2):
There exists an integer k such that for all $l \geq k$, each of these systems $\left(I^{(l)}, \Omega^{(k)}\right)$ is involutive. Furthermore, if $M^{(k)}$ is empty for some $k \geq 1$, then (I, Ω) has no n-dimensional integral manifolds.

Example (5):
Consider the system of partial differential equations of a single variable $u = u(x, y, z)$ defined on $\mathbb{R}^3$

$$u_{xx} = u_{yy} = u_{zz}. \qquad (21)$$

Following the general procedure for transformation P.D.E. into exterior differential equations, the space we are working with is hence formed by the following variables

- u (1 variable);
- x, y, z (3 variables);
- u_x, u_y, u_z (3 variables);
- $u_{xx} = u_{yy} = u_{zz},\ u_{xy} = u_{yx},\ u_{xz} = u_{zx},\ u_{yz} = u_{zy}$ (4 variables).

So the differential equation is defined on an 11-dimensional space formed by the variables above. The solution we seek for is a 3-dimensional integral manifold for which $dx \wedge dy \wedge dz \neq 0$, i.e., there are 8 variables that need to become dependent on x, y, z. The contact forms are

$$\begin{cases} \omega_u = du - u_x dx - u_y dy - u_z dz \\ \omega_x = du_x - u_{xx} dx - u_{xy} dy - u_{xz} dz \\ \omega_y = du_y - u_{xy} dx - u_{yy} dy - u_{yz} dz \\ \omega_z = du_z - u_{xz} dx - u_{yz} dy - u_{zz} dz \end{cases} \qquad (22)$$

which are set to zero. Recall that the zeros Cartan character s_0 is the number of the equations for which any linear integral elements of the system must be satisfied while ignoring the $dx, dy, dy,$ i.e., it is the rank of the matrix

$$\begin{pmatrix} & du & du_x & du_y & du_z & du_{xx} & du_{xy} & du_{yz} & du_{zx} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (23)$$

derived from the contact 1-forms, where the first row not part of the matrix. Obviously, this number is always equal to the number of independent 1-form equations above (since we are not allowed to have linear independence among the independence variables), and here we have $s_0 = 4$, even without forming the matrix explicitly.
Under exterior differentiation we have

$$\begin{cases} d\omega_u = -du_x \wedge dx - du_y \wedge dy - du_z \wedge dz \\ d\omega_x = -du_{xx} \wedge dx - du_{xy} \wedge dy - du_{xz} \wedge dz \\ d\omega_y = -du_{xy} \wedge dx - du_{yy} \wedge dy - du_{yz} \wedge dz \\ d\omega_z = -du_{xz} \wedge dx - du_{yz} \wedge dy - du_{zz} \wedge dz \end{cases} \qquad (24)$$

Then, we will use the equations $\omega_u = \omega_x = \omega_y = \omega_z = 0$, which give expressions for du, du_x, du_y, du_z to simplify these equations. Then, we get

$$\begin{cases} d\omega_u = 0 \\ d\omega_x = -du_{xx} \wedge dx - du_{xy} \wedge dy - du_{xz} \wedge dz \\ d\omega_y = -du_{xy} \wedge dx - du_{yy} \wedge dy - du_{yz} \wedge dz \\ d\omega_z = -du_{xz} \wedge dx - du_{yz} \wedge dy - du_{zz} \wedge dz \end{cases} \qquad (25)$$

Therefore s_1, we give dx, dy, dz the values $\varepsilon_1 x, \varepsilon_1 y,$ and $\varepsilon_1 z$ respectively, and $s_0 + s_1$ the rank of the matrix

$$\begin{pmatrix} du & du_x & du_y & du_z & du_{xx} & du_{xy} & du_{yz} & du_{zx} \\ \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_1 x & \varepsilon_1 y & 0 & \varepsilon_1 z \\ 0 & 0 & 0 & 0 & \varepsilon_1 y & \varepsilon_1 x & \varepsilon_1 z & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_1 z & 0 & \varepsilon_1 y & \varepsilon_1 x \end{pmatrix} \quad (26)$$

and we have $s_0 + s_1 = 7$, hence $s_1 = 3$. Then, for s_2, we form the matrix

$$\begin{pmatrix} du & du_x & du_y & du_z & du_{xx} & du_{xy} & du_{yz} & du_{zx} \\ \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_1 x & \varepsilon_1 y & 0 & \varepsilon_1 z \\ 0 & 0 & 0 & 0 & \varepsilon_1 y & \varepsilon_1 x & \varepsilon_1 z & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_1 z & 0 & \varepsilon_1 y & \varepsilon_1 x \\ 0 & 0 & 0 & 0 & \varepsilon_2 x & \varepsilon_2 y & 0 & \varepsilon_2 z \\ 0 & 0 & 0 & 0 & \varepsilon_2 y & \varepsilon_2 x & \varepsilon_2 z & 0 \\ 0 & 0 & 0 & 0 & \varepsilon_2 z & 0 & \varepsilon_2 y & \varepsilon_2 x \end{pmatrix} \quad (27)$$

The rank is 8, so $s_2 = 1$. Note that the matrix has already attained its maximal rank, so $s_3 = 0$. Obviously if we remove the first four columns, which can be non-zero only for the first four rows since we must enforce the 1-form equations, then we can calculate more easily the numbers $s_1, s_1 + s_2$, etc., which correspond to the ranks of the series of the matrices stacked together.

Then, we can apply Cartan's test. This corresponds to sitting$\varepsilon_1 y = \varepsilon_1 z = \varepsilon_2 x = \varepsilon_2 z = 0$ above, and then the characters can be read of directly from the 2-form equations as

$$s_1 = 3(du_{xx}, du_{xy}, du_{xz}), \qquad s_2 = 1(du_{yz}), \qquad s_3 = 0. \quad (28)$$

$s_1$corresponding to the independent forms which multiply dx, shown in parentheses, etc. Note that this shortcut works also when we have higher order forms in our equations as long as the differentials of dependent variables enter only linearly, i.e., we don't have terms such as

$$du_{xx} \wedge du, \qquad dx \wedge dy \wedge du_x \wedge du_y.$$

If such forms are present, we cannot use this shortcut and the calculation of even the reduced characters become very difficult, since first we need to find the general 3-dimensional linear elements, which is already more difficult since now there would be quadratic or higher order relations among the parameters, and then calculate using the elements, the steps of calculation required being roughly quadratic in the total number of variables. Actually in such a case, unless the number of variables is exceedingly small, a better way to proceed is to immediately effect a prolongation so as to get rid of all of the original higher order form equations, and the new systems is guaranteed to include only linear forms in the dependent variables.

We want to see the free parameters in an integral element

$$\begin{cases} du_{xx} = u_{xxx}dx + u_{xxy}dy + u_{xxz}dz \\ du_{xy} = u_{xyx}dx + u_{xyy}dy + u_{xyz}dz \\ du_{xz} = u_{xzx}dx + u_{xzy}dy + u_{xzz}dz \\ du_{yz} = u_{yzx}dx + u_{yzy}dy + u_{yzz}dz \end{cases} \quad (29)$$

The above expression holds 12 parameters. Substituting this back to the two form equations, we see that the free parameters are

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$$\begin{cases} u_{xxx} = u_{xyy} = u_{xzz} \\ u_{xxy} = u_{xyx} = u_{yzz} \\ u_{xxz} = u_{xzx} = u_{yzy} \\ u_{xyz} = u_{xzy} = u_{yzx} \end{cases} \quad (30)$$

so here the numbers of free parameters (N = 4), actually, even this substitution is unnecessary, since it is obvious that the free parameters are just the independent third order partial derivatives ofu. We have

$$N = 4 < s_1 + 2s_2 + 3s_3 = 5, \quad (31)$$

So Cartans test fails, the system is not involutive and prolongation is necessary. We can also see where things could go wrong as we laboriously wrote down the matrices: the real characters would correspond to matrices whose top-row labels include also dx, dy, dz. Now already at s_2, the rank of the reduced polar matrix is already constrained by the number of columns, and if we have more columns the rank could grow further, and consequently imply linear dependence among dx, dy, dz, which at the same time will imply the existence of further constraints on the free parameters in the integral element than implied by the counting of reduced Cartan characters. If this happens, which is the present case, it shows that the reduced character and real characters are not equal, and we are not in the involutive case. Prolongation corresponds, on the other hand, adding to the labels $du_{xx}, du_{xy}, du_{xz}, du_{yz}$, so we will not be constrained by the number of columns so soon.

Now, for prolongation we take $u_{xxx}, u_{xxy}, u_{xxz}, u_{xyz}$to be the new variables, adjoining (29) to the list of one-form equations (hence for the prolonged system, $s_0 = 4 + 4 = 8$), and we need to differentiate (29) to get some new two forms equations (the original one are now all identities). We now have $11 + 4 = 15$variables, and the number of variables that we want to get rid of is 12. We have

$$\begin{cases} d^2u_{xx} = du_{xxx} \wedge dx + du_{xxy} \wedge dy + du_{xxz} \wedge dz \\ d^2u_{xy} = du_{xxy} \wedge dx + du_{xxx} \wedge dy + du_{xyz} \wedge dz \\ d^2u_{xz} = du_{xxz} \wedge dx + du_{xyz} \wedge dy + du_{xxx} \wedge dz \\ d^2u_{yz} = du_{xyz} \wedge dx + du_{xxz} \wedge dy + du_{xxy} \wedge dz \end{cases} \quad (32)$$

For this new system, the reduced characters are

$$s_1 = 4\,(du_{xxx}, du_{xxy}, du_{xxz}, du_{xyz}), \qquad s_2 = 0, \qquad s_3 = 0. \quad (33)$$

So $s_0 + s_1 + s_2 + s_3 = 12$, the number of dependent variables, as it should be.

For completeness, we give the polar matrix for calculating s_2 of which we have removed the columns corresponding to $du, du_x, du_y, du_z, du_{xx}, du_{xy}, du_{xz}, du_{yz}$

$$\begin{pmatrix} du_{xxx} & du_{xxy} & du_{xxz} & du_{xyz} \\ \varepsilon_1 x & \varepsilon_1 y & \varepsilon_1 z & 0 \\ \varepsilon_1 y & \varepsilon_1 x & 0 & \varepsilon_1 z \\ \varepsilon_1 z & 0 & \varepsilon_1 x & \varepsilon_1 y \\ 0 & \varepsilon_1 z & \varepsilon_1 y & \varepsilon_1 x \\ \varepsilon_2 x & \varepsilon_2 y & \varepsilon_2 z & 0 \\ \varepsilon_2 y & \varepsilon_2 x & 0 & \varepsilon_3 z \\ \varepsilon_2 z & 0 & \varepsilon_2 x & \varepsilon_3 y \\ 0 & \varepsilon_2 z & \varepsilon_2 y & \varepsilon_3 x \end{pmatrix} \quad (34)$$

For the parameters,

$$\begin{cases} du_{xxx} = u_{xxxx}dx + u_{xxxy}dy + u_{xxxz}dz \\ du_{xxy} = u_{xxyx}dx + u_{xxyy}dy + u_{xxyz}dz \\ du_{xxz} = u_{xxzx}dx + u_{xxzy}dy + u_{xxzz}dz \\ du_{xyz} = u_{xyzx}dx + u_{xyzy}dy + u_{xyzz}dz \end{cases} \quad (35)$$

again there are 12 of them. The free ones can be algorithmically obtained by substituting these expressions into the two form equations, and we have

$$\begin{cases} u_{xxxx} = u_{xxyy} = u_{xxzz} \\ u_{xxxy} = u_{xxyx} = u_{xyzz} \\ u_{xxxz} = u_{xxzx} = u_{xyzy} \\ u_{xxyz} = u_{xxzy} = u_{xyzx} \end{cases} \quad (36)$$

so the number of free parameters is $N = 4$. Again, this substitution can be avoided by noting that the free parameters are just the independent fourth order partial derivatives of u. Now

$$N = 4 = s_1 + 2s_2 + 3s_3, \quad (37)$$

So Cartans test is satisfied, the system is involutive, and the general solution of the differential equation depends on four functions of one variable, by the Cartan–Kähler theorem.

CONCLUSIONS

As a conclusion, in this paper we discussed the Cartan– Kähler theorem and Applications to exterior differential system. Our main tools, is the quantum mechanics of exterior differential system. We have shown that our results coincide with previous results obtained by several authors. Our last observation was that, we can use the prolongation tower of an exterior differential system with independence condition (I, Ω) on an n-dimensional manifold M is defined to solve differential equation depends on four functions of one variable, by the Cartan–Kähler theorem.

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